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On the solvability of bivariate Hermite–Birkhoff interpolation problems *

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Abstract: In a recent paper by Hack (1987), certain bivariate polynomial Hermite–Birkhoff interpolation problems are considered and sufficient conditions for unique solvability are obtained by applying a general result on interpolation with tensor-functionals. In this paper, using a matricial approach and with the guideline of a result due to Stenger (1968), these conditions are shown to be also necessary and expressed in terms of the unisolvence of certain univariate interpolation problems. The results are valid not only for the polynomial case but also for interpolation with bivariate functions built from univariate extended complete Tchebycheff (ECT) systems.

Keywords: Hermite–Birkhoff, interpolation, bivariate.

1. Introduction

In a recent paper [3], Hack considers the question of the unisolvence of a certain class of polynomial bivariate Hermite–Birkhoff interpolation problems. After giving a general result on interpolation in tensor-product spaces, he makes use of it to obtain sufficient conditions for unique solvability of those Hermite–Birkhoff problems. Two-dimensional incidence matrices, which are a generalization of those used in the one-dimensional case (see [4]), are introduced.

R.A. Lorentz [5] gave another proof of Hack's results by means of the computation of the determinant associated with the interpolation problem.

In Section 2 we describe the type of bivariate Hermite–Birkhoff interpolation problems to be considered. In Section 3 we give Stenger's results [8] on the solution of linear systems with coefficient matrices which are submatrices of a Kronecker-product of matrices. We present the main theorem of [8] in an adequate way to be applied to our problem, and then, in Section 4, we show that the conditions given by Hack are not only sufficient, but also necessary for the unisolvence of this type of problems (for the considered interpolation spaces). Let us remark that we will consider generalized bivariate polynomials, i.e., linear combinations of functions which are products of univariate functions belonging to extended complete Tchebycheff systems. As particular cases we have algebraic polynomials and the rational functions with prescribed poles studied in [1,2].

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2. The interpolation problem

Let us consider the set of bivariate functions

$$\{F_{ij}(x, y) = \phi_i(x)\psi_j(y) \mid i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, m\} \tag{2.1}$$

where $(\phi_0, \phi_1, \ldots, \phi_n)$ (respectively $(\psi_0, \psi_1, \ldots, \psi_m)$) is an extended complete Tchebycheff (ECT) system of univariate functions on a real interval G (respectively H).

Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of nonnegative integers and consider the set of interpolation data of Hermite–Birkhoff type given by

$$L_{ij}f = \left.\frac{\partial^{r_i + t_{ij}} f}{\partial x^{r_i}\, \partial y^{t_{ij}}}\right|_{(x_i, y_{ij})}, \quad (i, j) \in I \subset \mathbb{N}_0 \times \mathbb{N}_0, \tag{2.2}$$

where $x_i \in G$ (respectively $y_{ij} \in H$) are not necessarily different and $r_i \leqslant n$, $t_{ij} \leqslant m$.

We assume, without loss of generality, that the index set $I = \{(i, j)\}$ in (2.2) satisfies the following conditions (which can be always achieved after reordering the abscissas of the interpolation points (x_i, y_{ij})):

(1)

$$I = \{(i, j) \mid i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, k(i)\},$$

with

$$\begin{aligned} m = k(0) = k(1) = \cdots = k(i_0) > k(i_0 + 1) \\ = \cdots = k(i_1) > \cdots > k(i_{s-1} + 1) = \cdots = k(i_s) \end{aligned} \tag{2.3}$$

and

$$i_s = n. \tag{2.4}$$

(2) For any $(i, j), (h, l) \in I$ one has

$$i = h \iff x_i = x_h \text{ and } r_i = r_h. \tag{2.5}$$

We are concerned with the following *interpolation problem P*. Find

$$p \in V = \operatorname{span}\{F_{ij} \mid (i, j) \in I\},$$

such that

$$L_{ij}p = z_{ij} \quad \forall (i, j) \in I, \tag{2.6}$$

where the z_{ij}'s are given real numbers.

It is important to realize that for each $(i, j) \in I$,

$$L_{ij}(F_{hk}) = L_i(\phi_h) \cdot L_j^{(i)}(\psi_k), \tag{2.7}$$

with

$$L_i(\phi_h) = \left.\frac{\mathrm{d}^{r_i}\phi_h}{\mathrm{d}x^{r_i}}\right|_{x_i}, \qquad L_j^{(i)}(\psi_k) = \left.\frac{\mathrm{d}^{t_{ij}}\psi_k}{\mathrm{d}y^{t_{ij}}}\right|_{y_{ij}}.$$

We will find consider the set I lexicographically ordered.

3. Regularity of generalized Kronecker-product matrices

In [8], Stenger gave an algorithm for solving a linear system whose coefficient matrix is a submatrix of the Kronecker-product C of two matrices A and B. If

$$A = [a_{ij}]_{i,j=0,1,\ldots,n}, \qquad B = [b_{hk}]_{h,k=0,1,\ldots,m}, \tag{3.1}$$

one has

$$C = \begin{bmatrix} a_{00}B & a_{01}B & \cdots & a_{0m}B \\ a_{10}B & a_{11}B & \cdots & a_{1m}B \\ \vdots & \vdots & & \vdots \\ a_{m0}B & a_{m1}B & \cdots & a_{mm}B \end{bmatrix}, \tag{3.2}$$

where the blocks $a_{ij}B$ are matrices of order $m+1$.

The coefficient matrix of the linear system (2.6) is a submatrix of

$$D = \begin{bmatrix} L_0(\phi_0)B_0 & L_0(\phi_1)B_0 & \cdots & L_0(\phi_n)B_0 \\ L_1(\phi_0)B_1 & L_1(\phi_1)B_1 & \cdots & L_1(\phi_n)B_1 \\ L_n(\phi_0)B_n & L_n(\phi_1)B_n & \cdots & L_n(\phi_n)B_n \end{bmatrix}, \tag{3.3}$$

where

$$B_i = [b_{kl}^{(i)}]_{k,l=0,\ldots,m} = [L_k^{(i)}(\psi_l)]_{k,l=0,\ldots,m}, \tag{3.4}$$

that is, with a notation similar to (3.2),

$$D = \begin{bmatrix} a_{00}B_0 & a_{01}B_0 & \cdots & a_{0n}B_0 \\ a_{10}B_1 & a_{11}B_1 & \cdots & a_{1n}B_1 \\ \vdots & \vdots & & \vdots \\ a_{n0}B_n & a_{n1}B_n & \cdots & a_{nn}B_n \end{bmatrix}. \tag{3.5}$$

However, the reasonings and results of Stenger for (3.2) have obvious extensions to the case of matrices (3.5). Since our aim is to use those results to discuss the unisolvence of the problem P, we will give an appropriate reformulation of the main theorem of [8].

In the sequel, if J, T are ordered subsets of the index sets of rows and columns, respectively, of a matrix M, we denote by

$$M[J, T]$$

the matrix consisting of the elements of the rows J and columns T of M.

Let D be a matrix of type (3.5) with B_i given by (3.4). Denote

$$T_i = \{0, 1, \ldots, k(i)\}, \quad i = 0, 1, \ldots, n, \tag{3.6}$$

(we assume that (2.3), (2.4) hold),

$$S_i = \{(m+1)i, (m+1)i+1, \ldots, (m+1)i+k(i)\} \tag{3.7}$$

and

$$T = \bigcup_{i=0}^{n} S_i. \tag{3.8}$$

Finally, let us denote by J_p the set

$$J_p = \{0, 1, \dots, i_p\}, \tag{3.9}$$

with i_p, $p = 0, 1, \dots, s$, defined by (2.3).

Theorem 1. *If* (2.3), (2.4) *hold, the matrix* $D[T, T]$ *is regular if and only if the matrices*

$$A[J_p, J_p], \quad p = 0, 1, \dots, s, \tag{3.10}$$

and

$$B_i[T_i, T_i], \quad i = 0, 1, \dots, n, \tag{3.11}$$

are regular.

The proof is just that of the main theorem of [8] with obvious modifications, and it is based on the construction of an algorithm to solve linear systems with submatrices of D given by (3.5). More details can be found in [6].

4. Unisolvence of the bivariate interpolation problem

Let us return to the problem P. According to the construction of the index set I and taking into account (2.5), the coefficient matrix of the linear system (2.6) is

$$D[T, T],$$

with D, T defined by (3.3), (3.4), (3.7) and (3.8).

Consider the following interpolation problems.

Problem PX_p (*respectively* **Problem PY_i**). Find $g(x) \in \operatorname{span}\{\phi_0, \phi_1, \dots, \phi_{i_p}\}$ (respectively $h(y) \in \operatorname{span}\{\psi_0, \dots, \psi_{k(i)}\}$), such that

$$L_i g = \left.\frac{\mathrm{d}^{r_i} g(x)}{\mathrm{d}x^{r_i}}\right|_{x_i} = \alpha_i \qquad \forall i = 0, 1, \dots, i_p, \tag{4.1}$$

(respectively,

$$L_j^{(i)} h = \left.\frac{\mathrm{d}^{t_{ij}} h(y)}{\mathrm{d}y^{t_{ij}}}\right|_{y_{ij}} = \alpha_{ij} \qquad \forall j = 0, 1, \dots, k(i)), \tag{4.2}$$

where α_i, α_{ij} are given real numbers.

Both problems are univariate Hermite–Birkhoff interpolation problems with coefficient matrices in the linear systems (4.1), (4.2) given, respectively, by (3.10) and (3.11). Therefore, Theorem 1 can be reformulated in terms of interpolation as follows, showing that the conditions of [3] are not only sufficient but also necessary for the unique solvability of the considered problem.

Theorem 1′. *Under the hypothesis of Section 2, problem P has a unique solution if and only if all the problems PX_p, PY_i, $p = 0, 1, \ldots, s$, $i = 0, 1, \ldots, n$, have a unique solution.*

Example. Let us consider the polynomial interpolation problem P with data

$$f(a, u_0), \quad \frac{\partial f(a, u_1)}{\partial y}, \quad f(b, u_2), \quad \frac{\partial f(b, u_3)}{\partial y}, \quad \frac{\partial f(b, u_4)}{\partial x}, \quad \frac{\partial f(c, u_5)}{\partial x},$$

where a, b, c are *different* real numbers, and u_i, $i = 0, 1, \ldots, 5$, are not necessarily different.

We can reformulate the problem, according to Section 2, denoting

$$\begin{array}{llll} x_0 = a, & x_1 = b, & x_2 = b, & x_3 = c, \\ y_{00} = u_0, & y_{01} = u_1, & & \\ y_{10} = u_2, & y_{11} = u_3, & & \\ y_{20} = u_4, & & & \\ y_{30} = u_5. & & & \end{array}$$

In this case, we have $n = 3$, $k(0) = k(1) = 1$, $k(2) = k(3) = 0$, $i_0 = 1$, $i_1 = 3$ and $s = 1$. The interpolation space is span$\{1, y, x, xy, x^2, x^3\}$.

The problems PY_i are given by:

Problem	Data	Interpolation space
PY_0	$h(y_{00})$, $h'(y_{01})$	Univariate polynomial degree 1 in y
PY_1	$h(y_{10})$, $h'(y_{11})$	Univariate polynomial degree 1 in y
PY_2	$h(y_{20})$	Univariate polynomial degree 0 in y
PY_3	$h(y_{30})$	Univariate polynomial degree 0 in y

and therefore are all unisolvent.

The problems PX_p are:

Problem	Data	Space
PX_0	$g(x_0)$, $g(x_1)$	Univariate polynomial degree 1 in x
PX_1	$g(x_0)$, $g(x_1)$, $g'(x_2)$, $g'(x_3)$	Univariate polynomial degree 3 in x

Therefore, PX_0 being unisolvent, the unisolvence of problem P is equivalent to that of PX_1. Some simple calculations show that PX_1 has a unique solution if and only if

$$c \neq \tfrac{1}{3}(2a + b). \tag{4.3}$$

As an important particular case we have the problems where the following condition (in addition to those of Section 2) holds. If

$$\left.\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha\, \partial y^\beta}\right|_Q$$

is one of the interpolation data of the problem ($Q \in \mathbb{R}^2$), then every

$$\left.\frac{\partial^{\tau+\omega} f}{\partial x^\tau\, \partial y^\omega}\right|_Q,$$

with $0 \leqslant \tau \leqslant \alpha$, $0 \leqslant \omega \leqslant \beta$, must also be one of the interpolation data.

This type of problems has been studied in the last years by several authors (see [7] for results and references). A simple reasoning shows that problems PY_i and PX_p are, in this case, univariate Hermite problems, and therefore the problem P is unisolvent.

Finally, let us remark that taking advantage of the associativity of tensor-products, our results can be extended (with the usual notational difficulties) to higher dimensions.

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